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ACTUAL SIZE OF THE WELCH-ASPIN TEST FOR THE BEHRENS-FISHER PROBLEM

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ACTUAL SIZE OF THE WELCH-ASPIN TEST FOR THE BEHRENS-FISHER PROBLEM submitted by Gaysawn Jakes in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis studies the problem of finding the actual size of a Welch-Aspin test as applied to the solution of the well-known Behrens-Fisher problem. Methods of finding the actual size, due to Wang, and Mehta and Srinivasan, are discussed. Modifications to Wang's method are attempted, and the resulting methods are used to find the actual size for a larger number of parameter values than used in previous studies. Numerical results are given for each method. These results are compared, not only with each other, but also (where applicable) with those results obtained in earlier studies. It is seen that the modified method of Wang tends to produce a size of test in closer accordance with an assigned size of test α.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 The Problem

In the field of statistical data analysis, it is often necessary to compare two populations with respect to a certain characteristic. For example, agriculturalists may want to compare the yields of wheat from two different farms. Examples such as this form the basis of the Two Means Problem, which is defined as that of comparing the means of two normal populations.

Suppose that $\bar{x}_1$ is the mean, and S_1^2 the sample variance, of a sample of size n_1, taken from a normal population with true mean μ_1 and true variance σ_1^2 and that $\bar{x}_2$, S_2^2, n_2, μ_2, and σ_2^2 are the corresponding values from a second normal population.

If the exact values of the population variances σ_1^2 and σ_2^2 are known, then the problem of testing the hypothesis $\mu_1 = \mu_2$ can be solved using the Normal distribution. If the exact values are not known, but the ratio $\theta = \sigma_1^2 / \sigma_2^2$ is known, then this problem is solved

using the statistic

$$d = \frac{\bar{x}_1 - \bar{x}_2}{\left\{ \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \right\}^{\frac{1}{2}}} \Bigg/ \left\{ \frac{\frac{(n_1-1)s_1^2}{\sigma_1^2} + \frac{(n_2-1)s_2^2}{\sigma_2^2}}{n_1 + n_2 - 2} \right\}^{\frac{1}{2}}$$

which has the t-distribution with $n_1 + n_2 - 2$ degrees of freedom if $\mu_1 = \mu_2$.

If, however, the value of θ is unknown, then the above d statistic cannot be calculated: the solving of the two-means problem when θ is indeed unknown, is known as the Behrens-Fisher problem.

1.2 Review of the Literature

The solving of the Behrens-Fisher problem has been the subject of a great deal of research. It was first considered by Behrens (1929), who suggested that the distribution of the difference between two means could be expressed in terms of observations in the samples from the two normal populations. Fisher (1936) extended this as the correct "exact" solution and Sukhatme (1938) drew up tables for the distribution of the Behrens-Fisher statistic. Bartlett (1936, 1939), among others, criticized this solution by saying that the probability of rejecting the hypothesis of the equality of the two means (using Fisher's

method) was generally less than the assigned size of the test, and tried to solve the problem using the theory of confidence intervals. Welch (1938) briefly mentioned this method, and Scheffé (1943) wrote up the full solution: this solution was based on the t-distribution, and was proven to be the best solution of its type available.

Welch (1947) developed a solution based on a series approximation for a critical value $V_\alpha(c)$, such that

$$Pr\left\{ V \geq V_\alpha(c) \right\} = \alpha$$

where

$$c = \frac{\frac{S_1^2}{n_1}}{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}$$

and

$$V = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\left\{ \frac{S_1^2}{n_1} + \frac{S_2^2}{n_2} \right\}^{\frac{1}{2}}}$$

is the statistic, and

$$\alpha = \text{assigned size of test}$$

Aspin (1948), as well as Trickett, Welch and James (1956), tabulated critical values $V_\alpha(c)$ for this solution for certain sets of values α, n_1, n_2 and c. Other solutions have been proposed by Wald (1955), Banerjee (1961) and Pagurova (1968). Scheffé (1970) provided a recent paper that reviewed many of the above solutions.

1.3 Objectives of the Research

In the paper of Scheffé (1970), the question is raised as to the actual size of the test for Welch's solution. Welch, himself, in an appendix to Aspin (1949) found the actual size for a few values of the parameters ($\alpha = .05$, $n_1 = n_2 = 7$, $\theta = \frac{1}{9}, \frac{1}{4}, \frac{3}{7}, \frac{2}{3}, 1$), using an unspecified method. Mehta and Srinivasan (1970), while comparing the actual size of the test for various solutions, used a method of triple integration developed by Golhar (1964). Wang (1971) found the actual size of the Welch test for selected sets of parameter values by using a method involving numerical quadrature and the t-distribution. However, all of these writers presented results for only a few sets of α, n_1 and n_2. Therefore, it is the objective of this research to investigate these methods for the purpose of determining the actual size of the test for a greater number of sets of parameter values.

CHAPTER II

WANG'S METHOD

In the paper of Wang (1971), it is stated that for a given test with statistic $w = \frac{(x - \mu)}{S}$, and critical value C_α (c) at an assigned size of test α, the actual size of the test may be written as

$$P\ (C_\alpha, \theta) = Pr\left(\frac{x - \mu}{S} \geq C_\alpha\ (c) \mid \theta \right)$$

where

$$x = \bar{x}_1 - \bar{x}_2$$

$$\mu = \mu_1 - \mu_2$$

$$S = \left\{ \frac{S_1^2}{n_1} + \frac{S_2^2}{n_2} \right\}^{\frac{1}{2}}$$

and

$$\theta = \frac{\sigma_1^2}{\sigma_2^2}$$

Multiplying both sides of the inequality by the term

$$\left\{ \frac{s^2}{\dfrac{\sigma^2}{f_1+f_2}\left(\dfrac{f_1 s_1^{\,2}}{\sigma_1^{\,2}} + \dfrac{f_2 s_2^{\,2}}{\sigma_2^{\,2}}\right)} \right\}^{\frac{1}{2}}$$

where

$$\sigma^2 = \frac{\sigma_1^{\,2}}{n_1} + \frac{\sigma_2^{\,2}}{n_2}$$

the above expression becomes

$$P\,(C_\alpha,\theta) = Pr \left\{ (x - \mu) \sqrt{\frac{f_1 + f_2}{\left(\dfrac{f_1 s_1^{\,2}}{\sigma_1^{\,2}} + \dfrac{f_2 s_2^{\,2}}{\sigma_2^{\,2}}\right)\sigma^2}} \geq \right.$$

$$\left. C_\alpha(c) \cdot \sqrt{\frac{\sigma_1^{\,2}\sigma_2^{\,2}\ (f_1+1)\ (f_2+1)\ (f_1+f_2)\ (1+r)}{[\sigma_1^{\,2}(f_2+1) + \sigma_2^{\,2}(f_1+1)]\ [rf_1(f_1+1)\sigma_2^{\,2} + f_2(f_2+1)\sigma_1^{\,2}]}}\ \middle|\ \theta \right\} \tag{2.1}$$

where

$$C_\alpha(c) = \text{Welch-Aspin critical value}$$

$$r = \frac{s_1^{\,2}(f_2+1)}{s_2^{\,2}(f_1+1)}$$

$$f_1 = n_1 - 1$$

$$f_2 = n_2 - 1$$

$$c = \frac{\dfrac{S_1^2}{n_1}}{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}} \quad \text{or} \quad \frac{r}{1 + r}$$

Using the hypothesis that $\mu_1 = \mu_2$, the left-hand side of the inequality in the above expression is a Student's-t variable with $f_1 + f_2$ degrees of freedom, while the right hand side is a function of the random variable r and is distributed independently of t.

Wang defines a Beta $(\frac{f_1}{2}, \frac{f_2}{2})$ variable z as

$$z = \frac{\dfrac{S_1^2 f_1}{\sigma_1^2}}{\dfrac{S_1^2 f_1}{\sigma_1^2} + \dfrac{S_2^2 f_2}{\sigma_2^2}}$$

and relates this to r by using

$$z = \frac{1}{1 + \dfrac{S_2^2 f_2 \sigma_1^2}{S_1^2 f_1 \sigma_2^2}}$$

$$= \frac{1}{1 + \dfrac{\sigma_1^2 f_2 (f_2+1)}{r\sigma_2^2 f_1 (f_1+1)}}$$

This is equivalent to saying

$$r = \frac{z \ f_2(f_2+1) \ \sigma_1^2}{(1-z)f_1(f_1+1) \ \sigma_2^2} \qquad (2.2)$$

The right-hand side of the inequality (2.1) is thus labelled as a function of the Beta variable z, say h(z), and the complete expression (2.1) is written as an expectation over the Beta variable z

i.e.

$$P\ (C_\alpha, \theta) = E_z \left\{ Pr \left[t_{f_1+f_2} \geq h(z) \mid \theta, z \right] \right\}$$

$$= E_z \left\{ Q_{f_1+f_2} \left[h(z) \right] \right\}$$

where $Q_{f_1+f_2}[h(z)]$ is the upper tail of the t-integral (to the right of the point h(z)) with f_1+f_2 degrees of freedom.

The value of the actual size of the test can then be calculated by quadrature using Simpson's Rule in the following form:

$$P\ (C_\alpha, \theta) = E(Q_{f_1+f_2}\left[h(z)\right]) = \frac{\Delta}{3} \sum_{i=0}^{2n} w_i Q_{f_1+f_2}[h(z_i)]dz_i$$

$2n$ = number of intervals to be used in Simpson's Rule.

$z_i = \dfrac{i}{2n}$ (note that $0 \leq z_i \leq 1$ as z_i is a Beta variable)

$\Delta = \frac{1}{2n}$ (the length of each interval)

$w_i = 1, 4, 2, 4, 2, \cdots, 4, 1$

$dz_i =$ density function of the Beta variable z_i

which is

$$\frac{\Gamma\left(\frac{f_1}{2} + \frac{f_2}{2}\right)}{\Gamma\left(\frac{f_1}{2}\right)\,\Gamma\left(\frac{f_2}{2}\right)} \; z_i^{\frac{f_1}{2} - 1} \; (1 - z_i)^{\frac{f_2}{2} - 1}$$

An algorithm to evaluate this expression is given as follows:

1. Choose the values of α, n_1, n_2 and θ to be tested (there will be nine values of $\theta = \sigma_1^2/\sigma_2^2$ tested for each set of n_1, n_2 and α, namely $\frac{1}{9}, \frac{2}{8}, \frac{3}{7}, \frac{4}{6}, \frac{5}{5}, \frac{6}{4}, \frac{7}{3}, \frac{8}{2}, \frac{9}{1}$).

2. Fit a fourth-degree polynomial to the available 11 values in the particular row of a Welch-Aspin table of critical values corresponding to the choice of α, n_1 and n_2 (this polynomial will be used to find the value of $C_\alpha(c_i)$, the Welch critical value, necessary to calculate a particular value of $h(z_i)$).

3. Decide how many intervals will be used, and call this value 2n.

4. For each value of z_i ($i = 0$ to $2n$), calculate the value of $h(z_i)$ by transforming z_i to r_i using (2.2) and making the appropriate substitutions in the right-hand side of the inequality in (2.1). Use this value of $h(z_i)$ to determine the value of the t-integral to the right of the point $h(z_i)$ namely, $Q_{f_1+f_2}[h(z_i)]$.

5. Once all of the (2n+1) values of $Q_{f_1+f_2}[h(z_i)]$ have been determined, use Simpson's Rule in the form given earlier to evaluate $E_z\left\{ Q_{f_1+f_2}[h(z)] \right\}$, which is the desired estimate of the actual size of the test.

Two things should be mentioned here:

a) The fitting of the polynomial to the 11 points in a Welch-Aspin table of critical values corresponding to particular values of n_1, n_2 and α can be done using any accurate curve-fitting method. The polynomial should be fitted using the values of c indicated in the tables ($c = 0, .1, .2, \cdots, 1$), not r ($r = 0, \frac{1}{9}, \frac{2}{8}, \frac{3}{7}, \frac{4}{6}, \frac{5}{5}, \frac{6}{4}, \frac{7}{3}, \frac{8}{2}, \frac{9}{1}, \infty$) as was suggested by Wang. This is because the value of ∞ cannot be accurately represented on a curve.

b) The estimation for $Q_{f_1+f_2}[h(z_i)]$ can be done using the extremely accurate series approximation for the upper tail of the t-integral due to Hill (1970), which is described in some detail in Appendix A.

CHAPTER III

MEHTA AND SRINIVASAN'S METHOD

In the paper of Mehta and Srinivasan (1970), the Behrens-Fisher problem is defined in terms of testing the null hypothesis

$$H_0: \quad \eta = \frac{\mu_1 - \mu_2}{\sigma_1} = 0,$$

assuming that $\theta = \dfrac{\sigma_1^2}{\sigma_2^2}$ is unknown, and that (μ_1, σ_1^2) are the mean and variance of one normal population, with (μ_2, σ_2^2) being the mean and variance of a second normal population. The solution is expressed in the following general form.

Reject H_0 if

$$V = \frac{\bar{x}_1 - \bar{x}_2}{\left\{ \dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2} \right\}^{\frac{1}{2}}} \geq V_\alpha(c)$$

where $V_\alpha(c)$ is a function of

$$c = \frac{\dfrac{s_1^2}{n_1}}{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}$$

and $\bar{x}_1$, $\bar{x}_2$, are the means and s_1^2, s_2^2 the variances of samples of size n_1 and n_2 taken from the two normal populations.

For Welch's test, the critical value V_α (c) is expressed as (for an assigned size of test α)

$$\begin{aligned} V_\alpha(c) = \xi \Bigg[1 &+ \frac{1}{4f_1f_2}(1+\xi^2)\left\{f_2c^2+f_1(1-c)^2\right\} \\ &- \frac{1}{2f_1^2f_2^2}(1+\xi^2)\left\{f_2^2c^2+f_1^2(1-c)^2\right\} \\ &+ \frac{1}{3f_1^2f_2^2}(3+5\xi^2+\xi^4)\left\{f_2^2c^3+f_1^2(1-c)^3\right\} \\ &- \frac{1}{32f_1^2f_2^2}(15+32\xi^2+9\xi^4)\left\{f_2c^2+f_1(1-c)^2\right\}^2 \Bigg] \end{aligned} \tag{3.1}$$

where ξ is the $(1-\alpha)$ percentile of the standard normal distribution and $f_i = n_i - 1$ $(i = 1,2)$.

Golhar (1964) showed, by a series of laborious

calculations, that the probability

$$\beta(\eta, \theta) = \text{Prob}\left\{ V < V_\alpha(c) \right\}$$

is given by

$$\beta(\eta, \theta) = \frac{1}{\Gamma\left(\frac{f_1}{2}\right)\Gamma\left(\frac{f_2}{2}\right)} \int_0^\infty\int_0^\infty e^{-x-y} x^{\frac{1}{2}(f_1-2)} y^{\frac{1}{2}(f_2-2)}$$

$$\cdot \int_{-\infty}^{W(x,y)-\rho} \frac{1}{\sqrt{(2\pi)}} e^{\frac{-z^2}{2}} \, dxdydz \qquad (3.2)$$

where

$$W(x,y) = V_\alpha(\bar{c}) \sqrt{\left\{ \frac{2x\zeta}{f_1} + \frac{2y(1-\zeta)}{f_2} \right\}}$$

$$\bar{c} = \frac{1}{1 + \frac{y}{x}\frac{f_1}{f_2}\frac{(1-\zeta)}{\zeta}}$$

$V_\alpha(\bar{c})$ = equation (3.1) with c replaced by $\bar{c}$

$$\rho = \frac{\eta}{\left\{ \frac{1}{n_1} + \frac{1}{n_2\theta} \right\}^{\frac{1}{2}}}$$

$$\zeta = \frac{\frac{\sigma_1^2}{n_1}}{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

The power of the test is defined as the probability of rejecting H_0 when it is false, or in other words, Prob $\{V \geq V_\alpha(c)\}$ or $1-\beta(\eta,\theta)$. This power is equal to the actual size of the test when $\eta = 0$. Therefore, the actual size of the test is given by the value of $1-\beta(0,\theta)$. The calculation of this value necessitates the evaluation of the triple integral (3.2) with ρ replaced by 0.

The inner integral in (3.2) is simply the standard normal cumulative distribution function $\Phi(W(x,y))$, and can be evaluated by using the relationship:

$$\Phi(W(x,y)) = \frac{1}{2}\left(1 + \operatorname{erf}\,(W(x,y)/\sqrt{2})\right)$$

where

$$\operatorname{erf}\left(\frac{W(x,y)}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\frac{W(x,y)}{\sqrt{2}}} e^{-u^2}\,du,$$

is called the "error function."

Thus

$$1 + \operatorname{erf}\left(\frac{W(x,y)}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{\frac{W(x,y)}{\sqrt{2}}} e^{-u^2}\,du,$$

and therefore

$$\beta(0,\theta) = \frac{1}{2\,\Gamma(\frac{f_1}{2})\,\Gamma(\frac{f_2}{2})}\int_0^\infty\!\!\int_0^\infty e^{-x-y}\,x^{\frac{1}{2}(f_1-2)}\,y^{\frac{1}{2}(f_2-2)}$$

$$\left(1 + \text{erf } \frac{(W(x,y))}{\sqrt{2}}\right) \text{dxdy.} \tag{3.3}$$

This resulting double integral can be evaluated using Simpson's Rule, by making the following transformations on x and y in order to make the limits finite, and using the FORTRAN function ERF to evaluate the error function.

Let

$$x = \frac{s}{1-s}$$

then

$$dx = \frac{1}{(1-s)^2}\,ds$$

and let

$$y = \frac{s'}{1-s'}$$

so that

$$dy = \frac{1}{(1-s')^2}\,ds'$$

This transforms the integral to the following:

$$\beta(0,\theta) = \frac{1}{2\,\Gamma(\frac{f_1}{2})\,\Gamma(\frac{f_2}{2})} \int_0^1\!\!\int_0^1 \frac{e^{-\left(\frac{s}{1-s} + \frac{s'}{1-s'}\right)}}{(1-s)^2\,(1-s')^2}\left(\frac{s}{1-s}\right)^{\frac{1}{2}(f_1-2)}$$

$$\cdot \left(\frac{s'}{1-s'}\right)^{\frac{1}{2}(f_2-2)} \left(1+\mathrm{erf}\left(\frac{W\left(\frac{s}{1-s},\frac{s'}{1-s'}\right)}{\sqrt{2}}\right)\right) ds\ ds'$$

From examining this integral, we can see that it is undefined at $s=1$, or $s'=1$, and close examination of the formula for $W\left(\frac{s}{1-s}, \frac{s'}{1-s'}\right)$ will show that it is undefined at $s=0$.

Therefore, a small value of ε (say, 10^{-7}) is chosen, and the value of the integral is found using the limits

$$\int_{\varepsilon}^{1-\varepsilon} \int_{0}^{1-\varepsilon} f\,(s,\, s')\ ds\ ds'$$

The Simpson's rule form for the above integral is

$$\frac{hk}{9} \sum w_{ij}\ f\,(s,s')$$

where

$h = \frac{1-2\varepsilon}{M}$ (the length of each interval on the s axis)

$k = \frac{1-\varepsilon}{M}$ (the length of each interval on the s' axis)

$M =$ number of points along each axis

$$f\,(s,s') = \frac{e^{-\frac{s}{1-s}-\frac{s'}{1-s'}}}{(1-s)^2\ (1-s')^2}\left(\frac{s}{1-s}\right)^{\frac{1}{2}(f_1-2)}\left(\frac{s'}{1-s'}\right)^{\frac{1}{2}(f_2-2)}$$

$$\left(1+\text{erf}\left(\frac{W\left(\frac{s}{1-s},\ \frac{s'}{1-s'}\right)}{\sqrt{2}}\right)\right)$$

W_{ij} = weighting coefficients which can be found by the following multiplication table:

TABLE 3.1

TABLE OF WEIGHTED COEFFICIENTS FOR TWO-DIMENSIONAL SIMPSON'S RULE

	1	4	2	4	2	4	. . .	2	4	1
1	1	4	2	4	2	4	. . .	2	4	1
4	4	16	8	16	8	16	. . .	8	16	4
2	2	8	4	8	4	8	. . .	4	8	2
4	4	16	8	16	8	16	. . .	8	16	4
2										
4	.									.
⋮	.									.
2	.									.
4	4	16	8	16	8	16	. . .	8	16	4
1	1	4	2	4	2	4	. . .	2	4	1

The resulting value of $\beta(0,\theta)$ is then subtracted from 1 to get the actual size of the test.

CHAPTER IV

PROGRAM AND RESULTS

4.1 Wang's Method

For Wang's method, a program was written in FORTRAN, using the procedure as outlined in the algorithm given in Chapter II (a flowchart for this program is given in Fig. 4.1). As Wang's method involves direct use of the Welch-Aspin tabulated critical values, it was decided to find the actual size of the test as such for the same values of α (and all finite pairs of sample sizes n_1, n_2) used in setting up the Welch-Aspin tables, for values of $\frac{\theta}{1+\theta}$ of $(\cdot 1, \cdot 2, \cdot 3, \cdots, \cdot 9)$. The results are given in Table 4.1. Investigation of these results shows that in all cases, the tabulated size (to three decimal places) is equal to the assigned size of test α.

4.2 Mehta and Srinivasan's Method

For the Mehta and Srinivasan method, a program was written in FORTRAN using the procedure as outlined in Chapter III (a flowchart for this program is given in

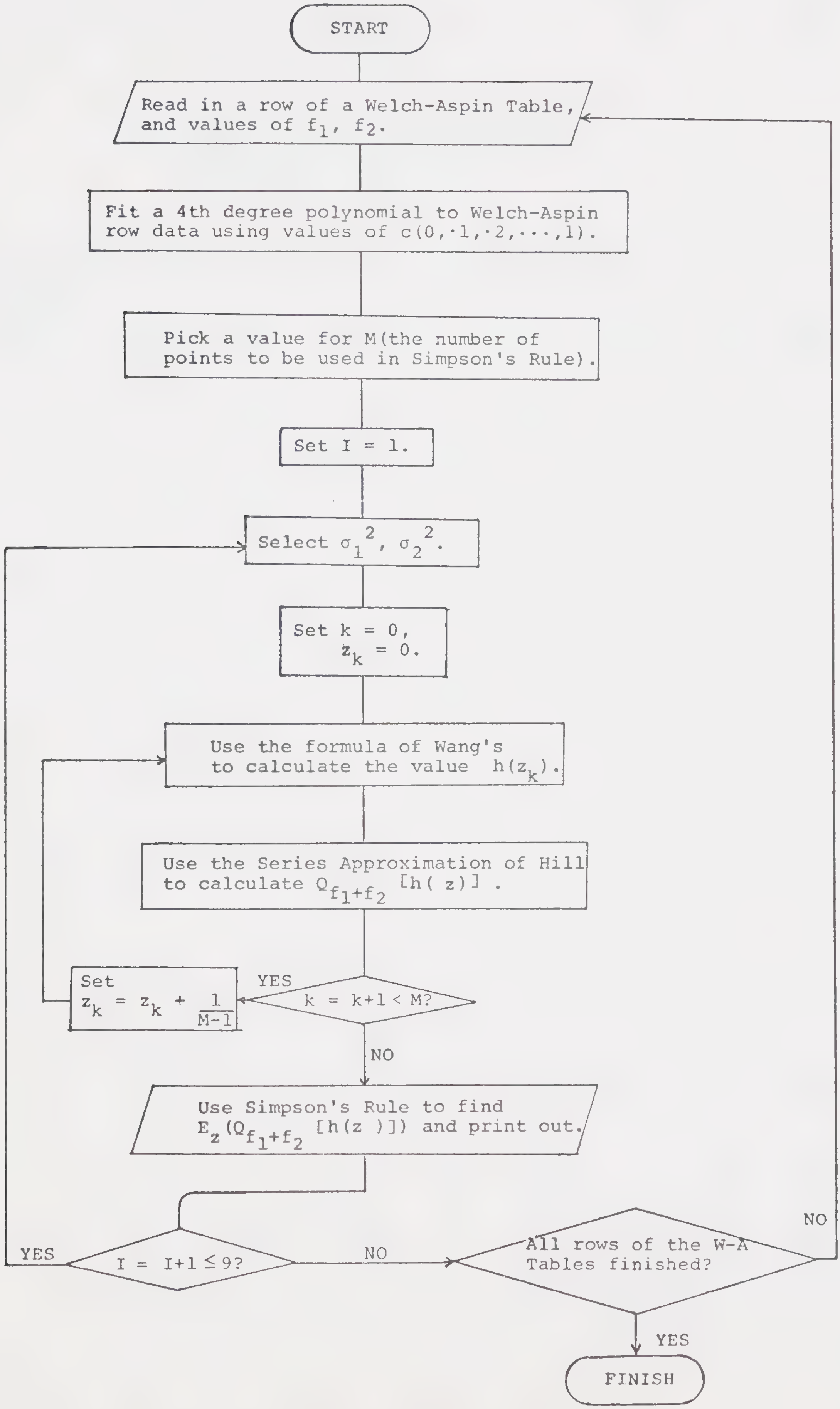

FIG. 4.1 FLOWCHART FOR WANG'S METHOD

TABLE 4.1

TABLE OF CALCULATED ACTUAL SIZES
USING WANG'S PROCEDURE
(α = .05)

	$\frac{\theta}{1+\theta}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
f_2= 6	f_1= 6	.0498	.0499	.0499	.0497	.0497	.0497	.0499	.0499	.0498
	8	.0498	.0500	.0500	.0500	.0499	.0499	.0500	.0501	.0500
	10	.0498	.0500	.0500	.0500	.0499	.0498	.0498	.0499	.0500
	15	.0499	.0499	.0500	.0500	.0500	.0500	.0500	.0501	.0502
	20	.0499	.0499	.0499	.0500	.0500	.0499	.0499	.0499	.0500
8	6	.0500	.0501	.0500	.0499	.0499	.0500	.0500	.0500	.0498
	8	.0501	.0502	.0501	.0500	.0500	.0500	.0501	.0502	.0501
	10	.0500	.0501	.0501	.0500	.0500	.0499	.0500	.0500	.0500
	15	.0500	.0501	.0501	.0500	.0500	.0499	.0499	.0500	.0501
	20	.0500	.0500	.0500	.0500	.0500	.0499	.0498	.0498	.0500
10	6	.0500	.0499	.0498	.0498	.0499	.0500	.0500	.0500	.0498
	8	.0500	.0500	.0500	.0499	.0500	.0500	.0501	.0501	.0500
	10	.0501	.0502	.0501	.0501	.0501	.0501	.0501	.0502	.0501
	15	.0501	.0501	.0501	.0500	.0500	.0500	.0500	.0500	.0501
	20	.0501	.0501	.0501	.0501	.0502	.0502	.0502	.0502	.0502
15	6	.0502	.0501	.0500	.0500	.0500	.0500	.0500	.0499	.0499
	8	.0501	.0500	.0499	.0499	.0500	.0500	.0501	.0501	.0500
	10	.0501	.0500	.0500	.0500	.0500	.0500	.0501	.0501	.0501
	15	.0502	.0501	.0500	.0500	.0500	.0500	.0500	.0501	.0502
	20	.0502	.0502	.0501	.0501	.0500	.0499	.0499	.0499	.0500
20	6	.0500	.0499	.0499	.0499	.0500	.0500	.0499	.0499	.0499
	8	.0500	.0498	.0498	.0499	.0500	.0500	.0500	.0500	.0500
	10	.0502	.0502	.0502	.0502	.0502	.0501	.0501	.0501	.0501
	15	.0500	.0499	.0499	.0499	.0500	.0501	.0501	.0502	.0502
	20	.0500	.0500	.0501	.0502	.0502	.0502	.0501	.0500	.0500

TABLE 4.1 (continued)

TABLE OF CALCULATED ACTUAL SIZES
USING WANG'S PROCEDURE
(α = .025)

	$\frac{\theta}{1+\theta}$	.1	.2	.3	.4	.5	.6	.7	.8	.9
f_2= 8	f_1= 8	.0250	.0252	.0251	.0251	.0250	.0251	.0251	.0252	.0250
	10	.0249	.0250	.0250	.0249	.0249	.0248	.0249	.0250	.0249
	12	.0249	.0250	.0250	.0250	.0249	.0249	.0249	.0250	.0251
	15	.0248	.0249	.0250	.0250	.0250	.0249	.0249	.0250	.0251
	20	.0248	.0249	.0250	.0250	.0250	.0250	.0250	.0250	.0250
10	8	.0249	.0250	.0249	.0248	.0249	.0249	.0250	.0250	.0249
	10	.0251	.0251	.0250	.0249	.0249	.0249	.0250	.0251	.0251
	12	.0251	.0252	.0252	.0251	.0251	.0250	.0250	.0251	.0250
	15	.0251	.0252	.0252	.0251	.0251	.0250	.0250	.0250	.0250
	20	.0251	.0252	.0252	.0251	.0250	.0249	.0248	.0249	.0250
12	8	.0251	.0250	.0249	.0249	.0249	.0250	.0250	.0250	.0249
	10	.0250	.0251	.0250	.0250	.0251	.0251	.0252	.0252	.0251
	12	.0250	.0250	.0250	.0250	.0249	.0250	.0250	.0250	.0250
	15	.0249	.0250	.0250	.0250	.0250	.0250	.0250	.0250	.0250
	20	.0250	.0250	.0250	.0250	.0250	.0250	.0250	.0251	.0251
15	8	.0251	.0250	.0249	.0249	.0250	.0250	.0250	.0249	.0248
	10	.0250	.0250	.0250	.0250	.0251	.0251	.0252	.0252	.0251
	12	.0250	.0250	.0250	.0250	.0250	.0250	.0250	.0250	.0249
	15	.0250	.0251	.0251	.0251	.0251	.0251	.0251	.0251	.0250
	20	.0250	.0250	.0249	.0249	.0248	.0249	.0249	.0250	.0250
20	8	.0250	.0250	.0250	.0250	.0250	.0250	.0250	.0249	.0248
	10	.0250	.0249	.0248	.0249	.0250	.0251	.0252	.0252	.0251
	12	.0251	.0251	.0250	.0250	.0250	.0250	.0250	.0250	.0250
	15	.0250	.0250	.0249	.0249	.0248	.0249	.0249	.0250	.0250
	20	.0250	.0250	.0250	.0249	.0249	.0249	.0250	.0250	.0250

TABLE 4.1 (continued)

TABLE OF CALCULATED ACTUAL SIZES
USING WANG'S PROCEDURE
(α = .01)

	$\frac{\theta}{1+\theta}$	.1	.2	.3	.4	.5	.6	.7	.8	.9
f_2=10	f_1=10	.0100	.0101	.0100	.0100	.0100	.0100	.0100	.0101	.0100
	12	.0100	.0101	.0100	.0100	.0099	.0099	.0100	.0100	.0101
	15	.0100	.0100	.0100	.0100	.0100	.0099	.0099	.0100	.0100
	20	.0100	.0100	.0100	.0100	.0100	.0099	.0099	.0099	.0100
	30	.0100	.0100	.0100	.0101	.0101	.0100	.0100	.0100	.0100
12	10	.0102	.0100	.0099	.0099	.0100	.0100	.0101	.0100	.0099
	12	.0100	.0101	.0101	.0100	.0100	.0100	.0101	.0101	.0100
	15	.0100	.0101	.0100	.0100	.0100	.0100	.0100	.0100	.0100
	20	.0100	.0100	.0100	.0100	.0099	.0099	.0099	.0099	.0100
	30	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100
15	10	.0100	.0100	.0099	.0099	.0100	.0100	.0100	.0100	.0100
	12	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0101	.0100
	15	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100
	20	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0101	.0101
	30	.0100	.0100	.0101	.0101	.0101	.0100	.0100	.0100	.0099
20	10	.0100	.0099	.0099	.0099	.0100	.0100	.0100	.0100	.0100
	12	.0100	.0099	.0099	.0099	.0099	.0100	.0100	.0100	.0100
	15	.0101	.0101	.0100	.0100	.0100	.0100	.0100	.0100	.0100
	20	.0101	.0101	.0100	.0100	.0100	.0100	.0100	.0101	.0101
	30	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0101
30	10	.0100	.0100	.0100	.0100	.0101	.0101	.0100	.0100	.0100
	12	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100
	15	.0099	.0100	.0100	.0100	.0101	.0101	.0101	.0100	.0100
	20	.0101	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100
	30	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100	.0100

TABLE 4.1 (continued)

TABLE OF CALCULATED ACTUAL SIZES
USING WANG'S PROCEDURE
(α =.005)

	$\frac{\theta}{1+\theta}$	.1	.2	.3	.4	.5	.6	.7	.8	.9
f_2=10	f_1=10	.0050	.0051	.0051	.0050	.0050	.0050	.0051	.0051	.0050
	12	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	15	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	20	.0049	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	30	.0049	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
12	10	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	12	.0050	.0051	.0051	.0050	.0050	.0050	.0051	.0051	.0050
	15	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	20	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	30	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
15	10	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	12	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	15	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	20	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	30	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
20	10	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0049
	12	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	15	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	20	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	30	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
30	10	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0049
	12	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	15	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	20	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050
	30	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050	.0050

Figure 4.2 - 4.3). It was decided to find the actual size of the test using the same values of α (and all finite pairs of sample sizes n_1, n_2) used in setting up the Welch-Aspin tables, for values of θ of (.0001, .001, .01, .1, 1, 10, 100, 1000, 10000) which Mehta and Srinivasan had used previously. In addition, values were computed for the particular case $\alpha = .05$, $n_1 = n_2 = 7$, and θ values of ($\frac{1}{9}$, $\frac{2}{8}$, $\frac{3}{7}$, $\frac{4}{6}$, $\frac{5}{5}$, $\frac{6}{4}$, $\frac{7}{3}$, $\frac{8}{2}$, $\frac{9}{1}$) in order to be able to compare the results of Mehta and Srinivasan's method, Wang's method, and Welch's unspecified method. The results for the general testing of Mehta and Srinivasan's method are given in Table 4.3, and the results for the special case are given in Table 4.4. Investigation of these results shows that in all but a few cases, the tabulated size (to three decimal places) is equal to the assigned size of test α. In all cases, the actual size to three decimal places is within ± 0.001 of the value of α.

It should be noted that, due to the limitations of the IBM 360/67 computer being used, results were not obtained for any set of parameters where f_1 was equal to 30. This is because the upper limit of s used (.9999999) was equivalent to x = 9999999, and the calculation of the term $x^{(f_1/2-1)}$ for these particular values of x and f_1 caused exponent overflow. Lowering of the upper limit of s to .99999 was tried, but this produced a highly inaccurate

result for the actual size of the test. For similar reasons, results for $f_2 = 30$ were also unobtainable. Therefore, in order to get an idea of how accurate the method actually is for values of f_1 (or f_2) = 30, another, more time consuming, version of the Mehta and Srinivasan method was implemented, as described in the following algorithm.

1. Decide on values of f_1, f_2, and α to be used.

2. Choose a value of $\theta = \frac{\sigma_1^2}{\sigma_2^2}$, and M (number of points).

3. Set NN equal to 1 and AVAL = 100.0.

4. Set the upper limit of the integral equal to 2^{NN}.

5. Use Simpson's Rule in two dimensions, to evaluate.

$$\text{SUM} = \int_{\varepsilon}^{2^{NN}} \int_{0}^{2^{NN}} e^{-x-y} \, x^{\frac{f_1}{2} - 1} \, y^{\frac{f_2}{2} - 1} \, (1+\text{erf}\,(W(x,y))\,dxdy.$$

using M points in each dimension.

6. If the absolute value of SUM-AVAL is less than

ε, then go to step 11. Else

7. Replace AVAL by the value of SUM.

8. Increase NN by 1.

9. Change the value of M to $2(M-1) + 1$.

10. Go to step 4.

11. Set the actual size of the test equal to

$$1 - \frac{\text{SUM}}{2\,\Gamma(\frac{f_1}{2})\,\Gamma(\frac{f_2}{2})}$$

12. If all values of theta have not been tested, then go to step 2. Else

13. Stop.

As computer implementation of this version of the Mehta and Srinivasan procedure is extremely time consuming and costly, it was decided to use the above algorithm for only one set of data ($f_1 = 10$, $f_2 = 30$, $\alpha = .01$). Results were as follows:

TABLE 4.2

CALCULATED SIZE OF TEST
(MEHTA AND SRINIVASAN PROCEDURE)
$\alpha = .01,\ f_1 = 10,\ f_2 = 30$

θ =	.0001	.001	.01	.1	1	10	100	1000	10000
actual size	.0100	.0100	.0100	.0100	.0102	.0102	.0101	.0101	.0101

It can be seen that these results are also in agreement (to three decimal places) with the value of the assigned size of test α. The author believes that similarly accurate results will occur for other sets of parameter values with one or both of n_i equalling 30.

4.3 Comparison of Results

Table 4.4 shows a comparison between the actual sizes of the test for $n_1 = n_2 = 7$, $\alpha = .05$, and values of θ of $(\frac{1}{9}, \frac{2}{8}, \frac{3}{7}, \ldots, \frac{8}{2}, \frac{9}{1})$, as calculated by Welch (1949), Wang (1970), and the present writer (using both Wang's and Mehta and Srinivasan's basic methods).

The results of Welch and the present writer

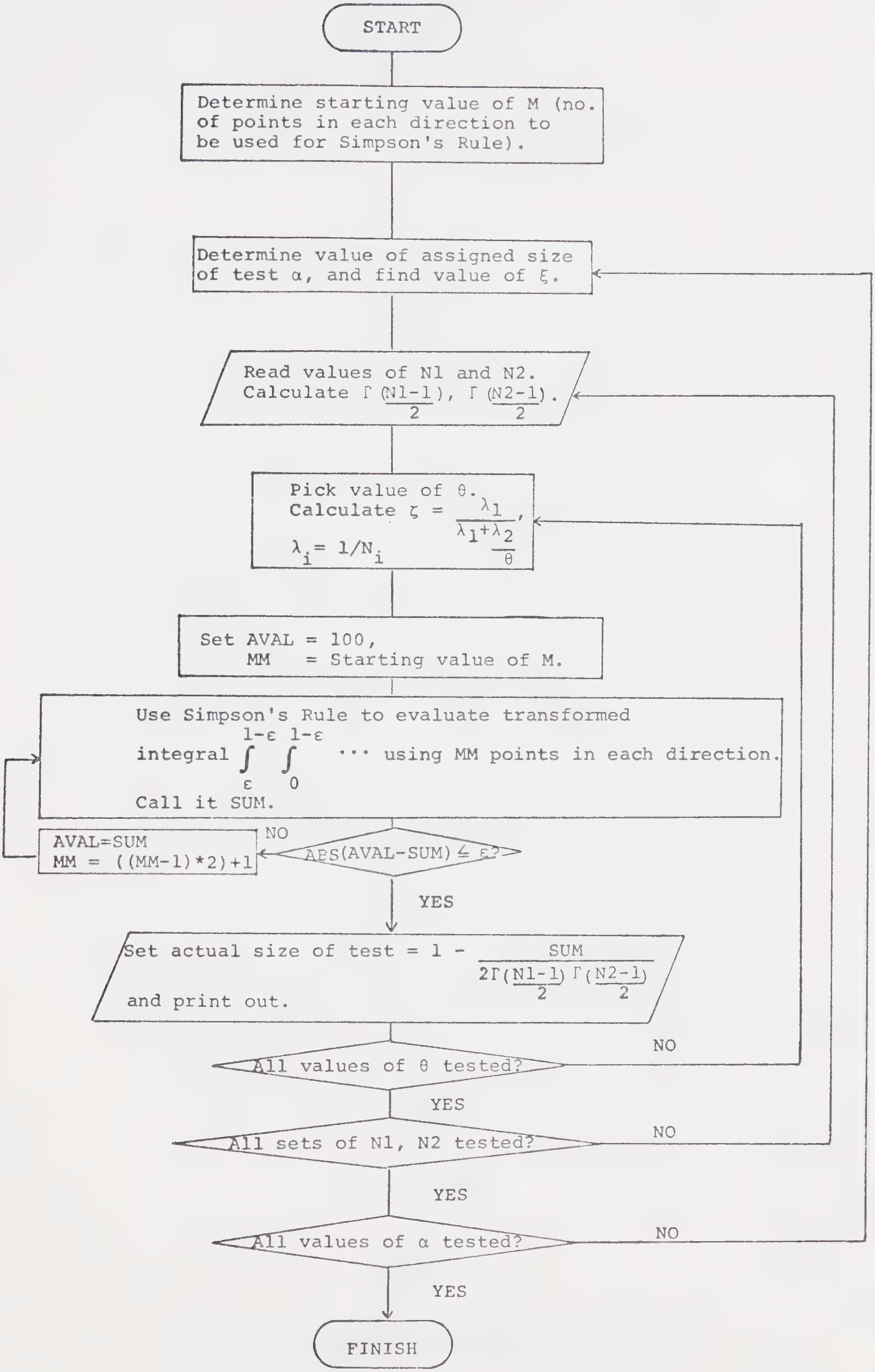

FIG. 4.2 FLOWCHART FOR MEHTA AND SRINIVASAN'S METHOD

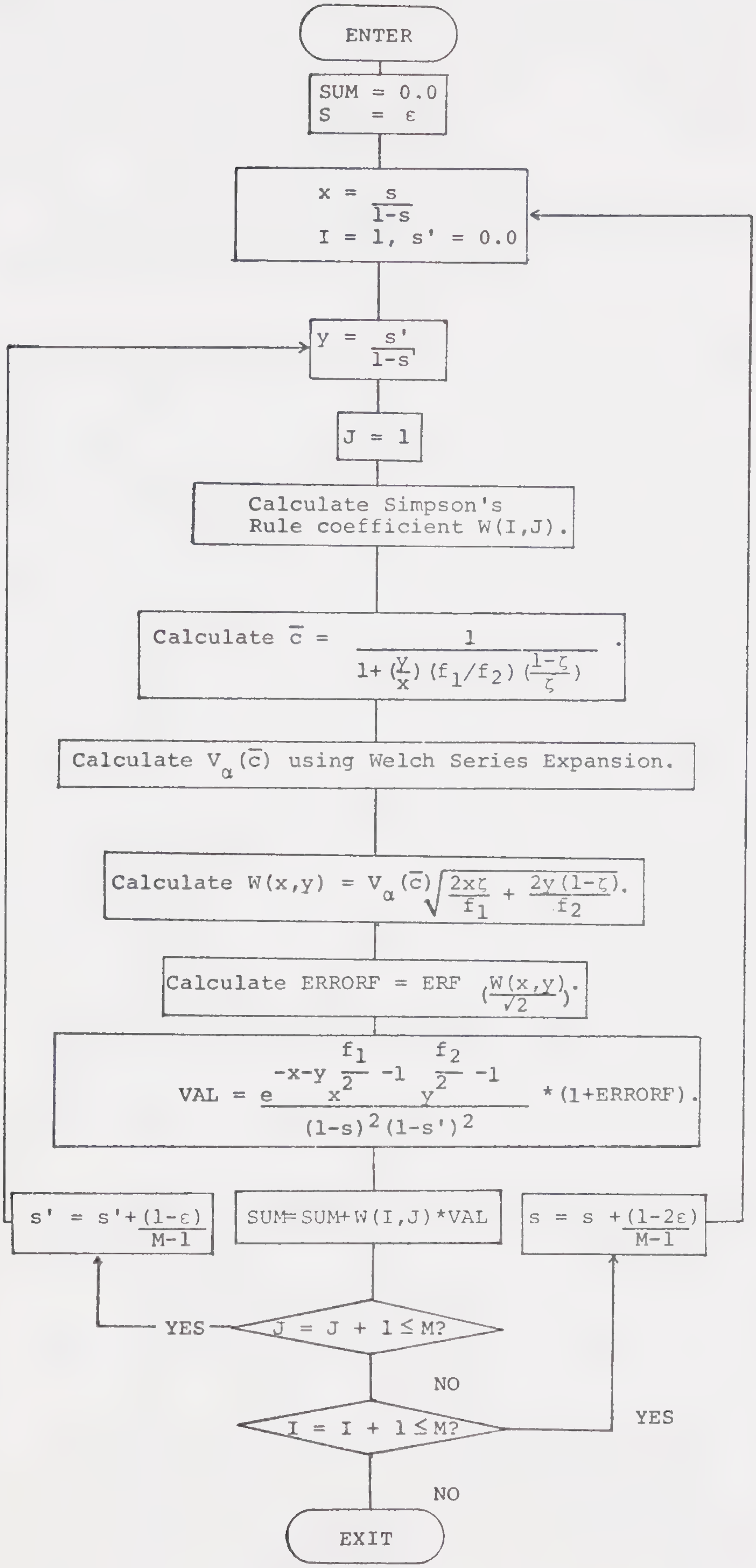

FIG. 4.3 FLOWCHART FOR SIMPSON'S RULE
(AS USED WITH MEHTA AND SRINIVASAN'S METHOD)

TABLE 4.3

TABLE OF CALCULATED ACTUAL SIZES
USING MEHTA AND SRINIVASAN'S PROCEDURE

(α = .05)

	θ	.0001	.001	.01	.1	1.0	10	100	1000	10000
f_2 = 6	f_1 = 6	.0503	.0503	.0503	.0505	.0493	.0505	.0503	.0503	.0503
	8	.0503	.0503	.0503	.0505	.0496	.0502	.0502	.0501	.0501
	10	.0503	.0503	.0503	.0505	.0498	.0501	.0501	.0501	.0501
	15	.0503	.0503	.0503	.0504	.0502	.0500	.0500	.0500	.0500
	20	.0505	.0505	.0505	.0505	.0505	.0500	.0501	.0501	.0501
8	6	.0501	.0501	.0502	.0502	.0496	.0505	.0503	.0503	.0503
	8	.0501	.0501	.0501	.0502	.0496	.0502	.0501	.0501	.0501
	10	.0501	.0501	.0501	.0502	.0497	.0501	.0501	.0501	.0501
	15	.0501	.0501	.0501	.0502	.0500	.0500	.0500	.0500	.0500
	20	.0503	.0503	.0503	.0503	.0502	.0501	.0501	.0501	.0501
10	6	.0501	.0501	.0501	.0501	.0498	.0505	.0503	.0503	.0503
	8	.0501	.0501	.0501	.0501	.0497	.0502	.0501	.0501	.0501
	10	.0501	.0501	.0501	.0501	.0498	.0501	.0501	.0501	.0501
	15	.0501	.0501	.0501	.0501	.0499	.0500	.0500	.0500	.0500
	20	.0502	.0502	.0502	.0502	.0501	.0501	.0501	.0501	.0501
15	6	.0500	.0500	.0500	.0500	.0502	.0504	.0503	.0503	.0503
	8	.0500	.0500	.0500	.0500	.0500	.0502	.0501	.0501	.0501
	10	.0500	.0500	.0500	.0500	.0499	.0501	.0501	.0501	.0501
	15	.0500	.0500	.0500	.0500	.0499	.0500	.0500	.0500	.0500
	20	.0502	.0502	.0502	.0502	.0501	.0502	.0501	.0501	.0501
20	6	.0501	.0501	.0501	.0500	.0505	.0505	.0505	.0505	.0505
	8	.0501	.0501	.0501	.0501	.0502	.0503	.0503	.0503	.0503
	10	.0501	.0501	.0501	.0501	.0501	.0502	.0502	.0502	.0502
	15	.0501	.0501	.0501	.0502	.0501	.0502	.0502	.0502	.0502
	20	.0503	.0503	.0503	.0503	.0502	.0503	.0503	.0503	.0503

TABLE 4.3 (continued)

TABLE OF CALCULATED ACTUAL SIZES
USING MEHTA AND SRINIVASAN'S PROCEDURE

(α = .025)

	θ	.0001	.001	.01	.1	1.0	10	100	1000	10000
f_2= 8	f_1= 8	.0252	.0252	.0252	.0253	.0246	.0253	.0252	.0252	.0252
	10	.0252	.0252	.0252	.0253	.0248	.0252	.0251	.0251	.0251
	12	.0252	.0252	.0252	.0253	.0249	.0251	.0251	.0251	.0251
	15	.0252	.0252	.0252	.0253	.0250	.0250	.0250	.0250	.0250
	20	.0253	.0253	.0253	.0254	.0253	.0251	.0252	.0252	.0252
10	8	.0251	.0251	.0251	.0252	.0248	.0253	.0252	.0252	.0252
	10	.0251	.0251	.0251	.0252	.0248	.0252	.0251	.0251	.0251
	12	.0251	.0251	.0251	.0252	.0248	.0251	.0251	.0251	.0251
	15	.0251	.0251	.0251	.0252	.0249	.0251	.0250	.0250	.0250
	20	.0252	.0252	.0252	.0253	.0252	.0252	.0252	.0252	.0252
12	8	.0251	.0251	.0251	.0251	.0249	.0253	.0252	.0252	.0252
	10	.0251	.0251	.0251	.0251	.0248	.0252	.0251	.0251	.0251
	12	.0251	.0251	.0251	.0251	.0249	.0251	.0251	.0251	.0251
	15	.0251	.0251	.0251	.0251	.0249	.0251	.0250	.0250	.0250
	20	.0252	.0252	.0252	.0252	.0251	.0252	.0252	.0252	.0252
15	8	.0250	.0250	.0250	.0250	.0250	.0253	.0252	.0252	.0252
	10	.0250	.0250	.0250	.0251	.0249	.0252	.0251	.0251	.0251
	12	.0250	.0250	.0250	.0251	.0249	.0251	.0251	.0251	.0251
	15	.0250	.0250	.0250	.0251	.0249	.0251	.0250	.0250	.0250
	20	.0252	.0252	.0252	.0252	.0251	.0252	.0252	.0252	.0252
20	8	.0252	.0252	.0252	.0251	.0253	.0254	.0253	.0253	.0253
	10	.0252	.0252	.0252	.0252	.0252	.0253	.0252	.0252	.0252
	12	.0252	.0252	.0252	.0252	.0251	.0252	.0252	.0252	.0252
	15	.0252	.0252	.0252	.0252	.0251	.0252	.0252	.0252	.0252
	20	.0253	.0253	.0253	.0253	.0252	.0253	.0253	.0253	.0253

TABLE 4.3 (continued)

TABLE OF CALCULATED ACTUAL SIZES
USING MEHTA AND SRINIVASAN'S PROCEDURE

(α = .01)

	θ	.0001	.001	.01	.1	1.0	10	100	1000	10000
f_2=10	f_1=10	.0101	.0101	.0101	.0102	.0098	.0102	.0101	.0101	.0101
	12	.0101	.0101	.0101	.0102	.0099	.0101	.0101	.0101	.0101
	15	.0101	.0101	.0101	.0102	.0100	.0101	.0101	.0101	.0101
	20	.0103	.0103	.0103	.0103	.0102	.0102	.0102	.0102	.0102
12	10	.0101	.0101	.0101	.0101	.0099	.0102	.0101	.0101	.0101
	12	.0101	.0101	.0101	.0101	.0099	.0101	.0101	.0101	.0101
	15	.0101	.0101	.0101	.0101	.0100	.0101	.0101	.0101	.0101
	20	.0102	.0102	.0102	.0102	.0102	.0102	.0102	.0102	.0102
15	10	.0101	.0101	.0101	.0101	.0100	.0102	.0101	.0101	.0101
	12	.0101	.0101	.0101	.0101	.0100	.0101	.0101	.0101	.0101
	15	.0101	.0101	.0101	.0101	.0100	.0101	.0101	.0101	.0101
	20	.0102	.0102	.0102	.0102	.0101	.0102	.0102	.0102	.0102
20	10	.0102	.0102	.0102	.0102	.0102	.0103	.0103	.0103	.0103
	12	.0102	.0102	.0102	.0102	.0102	.0102	.0102	.0102	.0102
	15	.0102	.0102	.0102	.0102	.0101	.0102	.0102	.0102	.0102
	20	.0103	.0103	.0103	.0103	.0103	.0103	.0103	.0103	.0103

TABLE 4.3 (continued)

TABLE OF CALCULATED ACTUAL SIZES
USING MEHTA AND SRINIVASAN'S PROCEDURE

(α = .005)

	θ	.0001	.001	.01	.1	1	10	100	1000	10000
f_2=10	f_1=10	.0051	.0051	.0051	.0052	.0049	.0052	.0051	.0051	.0051
	12	.0051	.0051	.0051	.0052	.0049	.0051	.0051	.0051	.0051
	15	.0051	.0051	.0051	.0052	.0050	.0051	.0050	.0050	.0050
	20	.0053	.0052	.0052	.0053	.0052	.0052	.0052	.0052	.0052
12	10	.0051	.0051	.0051	.0051	.0049	.0052	.0051	.0051	.0051
	12	.0051	.0051	.0051	.0051	.0049	.0051	.0051	.0051	.0051
	15	.0051	.0051	.0051	.0051	.0050	.0051	.0050	.0050	.0050
	20	.0052	.0052	.0052	.0052	.0052	.0052	.0052	.0052	.0052
15	10	.0050	.0050	.0050	.0051	.0050	.0052	.0051	.0051	.0051
	12	.0050	.0050	.0050	.0051	.0050	.0051	.0051	.0051	.0051
	15	.0051	.0051	.0051	.0051	.0050	.0051	.0051	.0051	.0051
	20	.0052	.0052	.0052	.0052	.0051	.0052	.0052	.0052	.0052
20	10	.0052	.0052	.0052	.0052	.0052	.0053	.0052	.0052	.0053
	12	.0052	.0052	.0052	.0052	.0052	.0052	.0052	.0052	.0052
	15	.0052	.0052	.0052	.0052	.0051	.0052	.0052	.0052	.0052
	20	.0053	.0053	.0053	.0053	.0053	.0053	.0053	.0053	.0053

TABLE 4.4

ACTUAL SIZES OF TESTS: COMPARISON OF RESULTS

$\frac{\theta}{1+\theta}$	Welch (1949)	Wang (1970)	Present	
			Applied Wang's	Applied Mehta's
.1	.0501	.0494	.0498	.0505
.2	.0500	.0495	.0499	.0502
.3	.0500	.0493	.0499	.0497
.4	.0498	.0492	.0497	.0494
.5	.0498	.0491	.0497	.0493
.6	.0498	.0492	.0497	.0494
.7	.0500	.0493	.0499	.0497
.8	.0500	.0495	.0499	.0502
.9	.0501	.0494	.0498	.0505

(Wang's method) are in close agreement (both with each other and with the assigned value of α). However, there is a slight discrepancy between these results and those of Wang and the present writer (Mehta and Srinivasan's method). The possible inaccuracy of Wang's results (1971) could be due to the fact that she may not have used as accurate an approximation (such as that of Hill (1970)) for the evaluation of the upper tail of the t integral. Any inaccuracies

of the results of the present author (Mehta and Srinivasan's method) are undoubtedly due to the way in which this procedure was implemented on the computer.

CHAPTER V

SUMMARY AND CONCLUSIONS

The methods of Wang and Mehta and Srinivasan for finding the actual size of a Welch-Aspin test as applied to the Behrens-Fisher problem, have been investigated in detail. Computer calculations of the actual size have been done for a large number of sets of parameters α, n_1, n_2 and θ. Of the two methods used, that of Wang was easier and more economical to implement on the computer for the following reason. Each step in Wang's method is clearly defined, and, in many cases, extremely accurate computer approximations exist to simplify the calculations involved. Good accurate computer approximations for the majority of steps in the Mehta and Srinivasan procedure do not exist (this is particularly true for the calculating of the double integral specified in the method): this means that long laborious calculations must be used. It is also true that the quantity $\beta(0,\theta)$ calculated in the Mehta-Srinivasan method covers a larger area than the quantity $E_z(Q_{f_1} + f_2 [h(z)])$ calculated in Wang's method. This is because the value of $\beta(0,\theta)$ must be subtracted from 1 to obtain the actual size of the test, which $E_z(Q_{f_1} + f_2 [h(z)])$

represents directly. There is a chance, therefore, that the calculating of $\beta(0,\theta)$ without accurate computer approximations may produce a less accurate answer than the calculating of $E_z(Q_{f_1+f_2}[h(z)])$. Perhaps because of this problem, the method of Wang generally produces answers in closer agreement with an assigned size of test α. It should be noted, however, that Mehta and Srinivasan's method does have the one advantage that it can be applied for any finite set of parameter values, while the use of Wang's method, as outlined, is restricted to those finite values of n_1, n_2 and α used in the calculating of the Welch-Aspin tables (Mehta and Srinivasan themselves quoted results for $f_1 = f_2 = 3$, $f_1 = f_2 = 19$, and $f_1 = 3$, $f_2 = 19$ - values not used by Aspin, Trickett, Welch or James in setting up their tables).

A recent paper of Golhar (1972) has just come to the present author's attention. In this paper, he suggests evaluating the Mehta and Srinivasan triple integral by using the error function approximation for the inner integral, and Gauss-Laguerre quadrature for the resulting double integral. Golhar gives results for the actual size of the test using this procedure (for a few select sets of f_1 and f_2), that appear to be in very close accordance with an assigned size of test α. The use of Gauss-Laguerre

quadrature appears to eliminate many of the problems encountered by the present author in implementing the Mehta and Srinivasan method on the computer. Therefore, it is recommended that this modification to the Mehta and Srinivasan method be used in any future investigations of the Welch-Aspin test. An attempt should be made at the same time to generalize Wang's method for all possible sets of finite parameter values.

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APPENDICES

APPENDIX A

SERIES APPROXIMATION FOR STUDENTS' T

A procedure for finding the value of the upper tail of Students' t distribution has been given by Hill (1970). This procedure involves four different approximations - the approximation to be used depends on the values of t and n. They are given as follows:

1. For large n ($n > 20$), or noninteger n, for all except large t, or for any $n > 200$, the series

$$s = x + (x^3 + 3x)/b -$$

$$(4x^7 + 33x^5 + 240x^3 + 855x)/(10b\ (b + 0.8x^4 + 100))$$

is evaluated using

$$x = \left(a.\mathrm{Ln}\ \left(1 + \frac{t^2}{n}\right)\right)^{\frac{1}{2}}$$

where

$$a = n - \frac{1}{2}$$

$$b = 48a^2$$

Then the upper tail of the t-integral (beyond the value t)

is given by the value of

$$\frac{1 - \mathrm{erf}\left(\frac{S}{\sqrt{2}}\right)}{2}$$

2. For small $n > 1$ and moderate t, with n being odd, the upper tail of the t-integral is given by the value of

$$\frac{1}{\pi}\left[\arctan (y) + \frac{y}{b}\left\{1 + \frac{2}{3b}\left\{\cdots \frac{n-5}{(n-4)b}\left\{1 + \frac{n-3}{(n-2)b}\right\}\cdots\right\}\right\}\right]$$

with

$$y = \left\{\frac{t^2}{n}\right\}^{\frac{1}{2}}$$

and

$$b = 1 + \frac{t^2}{n}$$

3. For small $n > 1$ and moderate t, with n being even, the upper tail of the t-integral is given by the value of

$$\frac{y}{2\sqrt{b}}\left\{1 + \frac{1}{2b}\left\{\cdots \frac{(n-5)}{(n-4)b}\left\{1 + \frac{(n-3)}{(n-2)b}\right\}\cdots\right\}\right\}$$

with y and b being defined as in 2.

4. For large t, the series

$$s = C(n)\, w^n \left\{ \frac{1}{n} + \frac{w^2}{2(n+2)} + \frac{(1)(3)\, w^4}{(2)(4)(n+4)} + \cdots \right\}$$

with

$$C(n) = \Gamma((n+1)/2)/(\sqrt{\pi}\, \Gamma(n/2)),$$

and

$$w = \left\{ 1 + \frac{t^2}{n} \right\}^{-\frac{1}{2}}$$

is summed until a negligible term occurs. The upper tail of the t-integral is then given by the value $\frac{1-s}{2}$.

The flowchart on the following page describes the computer program that is used to implement this procedure.

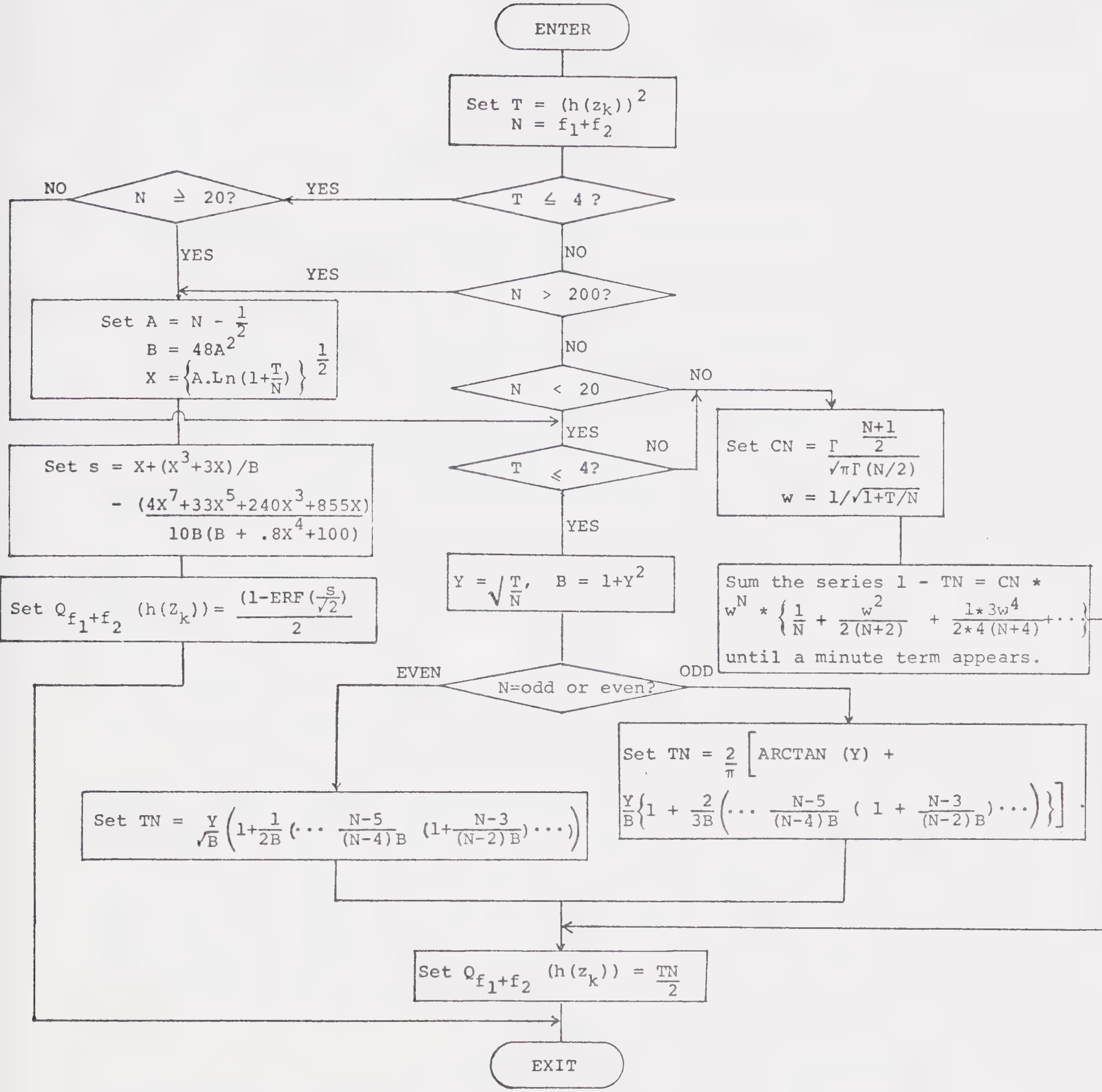

FIG. A1 FLOWCHART FOR HILL'S SERIES APPROXIMATION

APPENDIX B

A DESCRIPTION OF THE WELCH-ASPIN CRITICAL VALUE SERIES EXPANSION

Welch (1947) derived the series expansion for $V_\alpha(c)$ by first considering the general problem concerning η, a population parameter, estimated by an observed quantity y which is normally distributed with variance σ_y^2.

Let

$$\sigma_y^2 = \sum_{i=1}^{k} \lambda_i \sigma_i^2$$

where λ_i are known positive numbers and σ_i^2 are unknown variances. Suppose that S_i^2 are the estimated variances based on f_i degrees of freedom, so that the sampling distribution of S_i^2 is chi-square (distributed independently of each other and of y) which is

$$p(S_i^2)\, dS_i^2 = \frac{1}{\Gamma(\frac{1}{2}f_i)} \left\{ \frac{f_i S_i^2}{2\sigma_i^2} \right\}^{(\frac{f_i}{2}-1)} \exp\left[-\frac{1}{2} \frac{f_i S_i^2}{\sigma_i^2} \right] d\left(\frac{f_i S_i^2}{2\sigma_i^2} \right)$$

(B.1)

Consider a simple particular case with k = 2. Define n_1, n_2 as sample sizes,

$$\eta = \mu_1 - \mu_2,$$

$$y = \bar{x}_1 - \bar{x}_2,$$

$$\sigma_y^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2},$$

$$\lambda_1 = \frac{1}{n_1}, \quad \lambda_2 = \frac{1}{n_2},$$

$$f_1 = n_1 - 1, \quad f_2 = n_2 - 1.$$

The estimated values of σ_i^2 are

$$s_1^2 = \frac{\sum_{i=1}^{n_1} (x_{1i} - \bar{x}_1)^2}{f_1}$$

and

$$s_2^2 = \frac{\sum_{i=1}^{n_2} (x_{2i} - \bar{x}_2)^2}{f_2} .$$

The probability statements about y were made similar in character to those which W. S. Gosset derived for the mean of a single sample of n observations ('Student,' 1908).

Welch sought a quantity h to satisfy the equation

$$\Pr\left[(y-\eta) < h(S_1^2, S_2^2, \cdots, S_k^2, P)\right] = P \qquad \text{(B.2)}$$

It is clear that h, computable from the observations, must be a function of the individual variances S_i^2 and of P, where P is a given probability.

Let $j(S_1^2, S_2^2, \cdots, S_k^2, P)$ denote probability that $(y-\eta)$ is less than $h(S_1^2, S_2^2, \cdots, S_k^2, P)$, given S_i^2 $(i = 1, 2, \cdots, k)$. Since y is distributed independently of S_i^2, we have,

$$j(S_1^2, S_2^2, \cdots, S_k^2, P) = \int_{u=-\infty}^{\frac{h(S_1^2, S_2^2, \cdots S_k^2, P)}{\sqrt{(\Sigma\lambda_i\sigma_i^2)}}} \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}u^2} du$$

(B.3)

Welch calls this $I\left\{\frac{h(S_1^2, S_2^2, \cdots, S_k^2, P)}{\sqrt{(\Sigma\lambda_i\sigma_i^2)}}\right\}$, where I is the normal probability integral. The condition of equation (B.2) is then simply that, if $j(S_1^2, S_2^2, \cdots, S_k^2, P)$ is averaged over the probability distributions of S_i^2, the result will equal P. Thus,

$$E\left\{j(S_1^2, S_2^2, \cdots, S_k^2, P)\right\} = P.$$

or

$$\int_{S^2} \cdots \int j(S_1^2, S_2^2, \cdots, S_k^2, P) \prod_{i=1}^{k} p\,(S_i^2)\, d\, S_i^2 = P.$$

(B.4)

Expand $j(S_1^2, S_2^2, \cdots, S_k^2, P)$ about an origin $(\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2)$ in a Taylor expansion. Thus $j(S_1^2, S_2^2, \cdots, S_k^2, P) = \exp\left[\sum_i (S_i^2 - \sigma_i^2)\, \partial_i\right] j(w_1, w_2, \cdots, w_k, P)$,

(B.5)

it being understood that the exponential is to be expanded in a power series in ∂_i and that ∂_i is to be interpreted so that

$$\partial_i \quad j(w_1, w_2, \cdots, w_k, P) = \left[\frac{\partial}{\partial w_i} \quad j(w_1, w_2, \cdots, w_k, P)\right]_{w_j=\sigma_j^2}$$

By substituting (B.5) into (B.4) Welch obtains

$$\Theta\ j(w_1, w_2, \cdots, w_k, P) = P, \qquad \text{(B.6)}$$

where

$$\Theta = \prod_i \int \exp\left[(S_i^2 - \sigma_i^2)\, \partial_i \right] p(S_i^2)\; d S_i^2 . \tag{B.7}$$

Substitution into (B.7) from (B.1), yields

$$\Theta = \prod_i \left\{ 1 - \frac{2\sigma_i^2 \partial_i}{f_i} \right\}^{-\frac{1}{2}f_i} \exp\left[-\sigma_i^2 \partial_i \right]$$

$$= \exp\left\{ - \Sigma\sigma_i^2 \partial_i - \tfrac{1}{2}\Sigma\; f_i \log\left(1 - \frac{2\sigma_i^2 \partial_i}{f_i}\right) \right\}$$

$$= \exp\left\{ \Sigma \frac{\sigma_i^4 \partial_i^2}{f_i} + \frac{4}{3}\; \Sigma \frac{\sigma_i^6 \partial_i^3}{f_i^2} + 2\Sigma \frac{\sigma_i^8 \partial_i^4}{f_i^3} + \text{etc.} \right\}$$

$$= 1 + \Sigma \frac{\sigma_i^4 \partial_i^2}{f_i} + \left\{ \frac{4}{3}\; \Sigma \frac{\sigma_i^6 \partial_i^3}{f_i^2} + \frac{1}{2}\left(\Sigma \frac{\sigma_i^4 \partial_i^2}{f_i} \right)^2 \right\} + \text{etc.} \tag{B.8}$$

Substituting (B.3) into (B.6), he gets

$$\Theta\, I \left\{ \frac{h(w_1, w_2, \cdots, w_k, P)}{\sqrt{(\Sigma\; \lambda_i \sigma_i^2)}} \right\} = P \tag{B.9}$$

This is a condensed form of the solution to this problem. The operator Θ constitutes a direction to carry out the partial differentiations indicated by (B.8). w_j must then

be equated to σ_j^2. The solution of the resulting equation will give $h(\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2, P)$ and therefore $h(S_1^2, S_2^2, \cdots, S_k^2, P)$.

Welch then proceeds to develop a series solution by writing $h(w)$ for $h(w_1, w_2, \cdots, w_k, P)$ and ξ for the normal deviate such that $I(\xi) = P$, and then expanding $I\left\{\frac{h(w)}{\sqrt{(\Sigma\lambda_i\sigma_i^2)}}\right\}$ in a Taylor series about ξ as origin. Thus

$$I\left\{\frac{h(w)}{\sqrt{(\Sigma\lambda_i\sigma_i^2)}}\right\} = \exp\left[\left\{\frac{h(w)}{\sqrt{(\Sigma\lambda_i\sigma_i^2)}} - \xi\right\}D\right] I(v), \tag{B.10}$$

where the exponential is to be expanded in powers of D, and that these powers are to be interpreted so that

$$D^r \quad I(v) = \left[\frac{d^r}{dv^r} \quad I(v)\right]_{v = \xi}$$

Equation (B.9) then becomes

$$\Theta \exp\left[\left\{\frac{h(w)}{\sqrt{\Sigma(\lambda_i\sigma_i^2)}} - \xi\right\} D\right] I(v) = I(\xi) \tag{B.11}$$

Welch states that this may be solved by successive approximations.

The initial approximation is the large sample normal approximation

$$h_0(w) \quad = \quad \xi \sqrt{(\Sigma\lambda_i w_i)}$$

and he writes

$$h(w) \quad = \quad \xi \sqrt{(\Sigma\lambda_i w_i)} + h_1(w) + h_2(w) + \text{etc.},$$

where $h_1(w)$ includes terms of order $\frac{1}{f_i}$, $h_2(w)$ terms of order $\frac{1}{f_i^{\,2}}$ and so on. Welch treats terms of order $\frac{1}{f_i^{\,3}}$ as negligible. Then (B.11) gives

$$\Theta \exp\left[\left\{\frac{\xi\sqrt{(\Sigma\lambda_i w_i)}}{\sqrt{(\Sigma\lambda_i\sigma_i^{\,2})}} - \xi\right\} D\right] \exp\left[\left\{\frac{h_1(w)+h_2(w)}{\sqrt{(\Sigma\lambda_i\sigma_i^{\,2})}}\right\} D\right] I(v)$$

$$= \quad I(\xi) \qquad (B.12)$$

or using (B.8)

$$\left[\frac{h_1(\sigma^2)D}{\sqrt{(\Sigma\lambda_i\sigma_i^{\,2})}} + \Sigma\,\frac{\sigma_i^{\,4}\partial_i^{\,2}}{f_i}\exp\left(\xi D\left\{\sqrt{\frac{\Sigma\lambda_i w_i}{\Sigma\lambda_i\sigma_i^{\,2}}} - 1\right\}\right)\right] I(v)$$

$$+\left[\frac{h_2(\sigma^2)D}{\sqrt{(\Sigma\lambda_i\sigma_i^{\,2})}} + \frac{1}{2}\,\frac{h_1^{\,2}(\sigma^2)D^2}{\Sigma\lambda_i\sigma_i^{\,2}}\right.$$

$$+\,\Sigma\,\frac{\sigma_i^{\,4}\partial_i^{\,2}}{f_i}\exp\left(\xi D\left\{\sqrt{\frac{\Sigma\lambda_i w_i}{\Sigma\lambda_i\sigma_i^{\,2}}} - 1\right\}\right)\frac{h_1(w)}{\sqrt{(\Sigma\lambda_i\sigma_i^{\,2})}}$$

$$+\left\{\frac{4}{3}\Sigma\frac{\sigma_i^{6}\partial_i^{3}}{f_i^{2}}+\frac{1}{2}\left(\Sigma\frac{\sigma_i^{4}\partial_i^{2}}{f_i}\right)^{2}\right\}\exp\left(\xi D\left\{\sqrt{\frac{\Sigma\lambda_i w_i}{\Sigma\lambda_i\sigma_i^{2}}}-1\right\}\right)\Bigg]$$

$$.\,I\,(v)\equiv 0 \quad (B.13)$$

By setting the first order term equal to zero Welch obtains

$$h_1(\sigma^2) = \frac{\xi\,(1+\xi^2)}{4}\,\frac{\left(\Sigma\frac{\lambda_i^{2}\sigma_i^{4}}{f_i}\right)}{(\Sigma\lambda_i\sigma_i^{2})^{\frac{3}{2}}}$$

which may be substituted in the second order term which, when set equal to zero, will give $h_2\ (\sigma^2)$. To terms of order $\frac{1}{f_i^{2}}$ Welch's solution is

$$h(S^2) = \xi\sqrt{\left(\Sigma\lambda_i S_i^{2}\right)}\left[1+\frac{(1+\xi^2)}{4}\,\frac{\Sigma\frac{\lambda_i^{2}S_i^{4}}{f_i}}{(\Sigma\lambda_i S_i^{2})^{2}}-\frac{(1+\xi^2)}{2}\,\frac{\Sigma\frac{\lambda_i^{2}S_i^{4}}{f_i^{2}}}{(\Sigma\lambda_i S_i^{2})^{2}}\right.$$

$$\left.+\frac{(3+5\xi^2+\xi^4)}{3}\,\frac{\Sigma\frac{\lambda_i^{3}S_i^{6}}{f_i^{2}}}{(\Sigma\lambda_i S_i^{2})^{3}}-\frac{(15+32\xi^2+9\xi^4)}{32}\,\frac{\left(\Sigma\frac{\lambda_i^{2}S_i^{4}}{f_i}\right)^{2}}{(\Sigma\lambda_i S_i^{2})^{4}}\right]$$

(B.14)

Aspin (1948) extended this series to terms of order $\frac{1}{f_i^{4}}$ and, along with Trickett, Welch, James (1956), used

this modified series to calculate the tables on the following pages.

Table B1 *Test for comparisons involving two variances which must be separately estimated*

Upper 5% critical values of $v = \dfrac{(y-\eta)}{\sqrt{(\lambda_1 s_1^2 + \lambda_2 s_2^2)}}$

(i.e. upper 10% critical values of $|v|$*)**

f_2	f_1 \ $\dfrac{\lambda_1 s_1^2}{\lambda_1 s_1^2 + \lambda_2 s_2^2}$	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
6	6	1·94	1·90	1·85	1·80	1·76	1·74	1·76	1·80	1·85	1·90	1·94
	8	1·94	1·90	1·85	1·80	1·76	1·73	1·74	1·76	1·79	1·82	1·86
	10	1·94	1·90	1·85	1·80	1·76	1·73	1·73	1·74	1·76	1·78	1·81
	15	1·94	1·90	1·85	1·80	1·76	1·73	1·71	1·71	1·72	1·73	1·75
	20	1·94	1·90	1·85	1·80	1·76	1·73	1·71	1·70	1·70	1·71	1·72
	∞	1·94	1·90	1·85	1·80	1·76	1·72	1·69	1·67	1·66	1·65	1·64
8	6	1·86	1·82	1·79	1·76	1·74	1·73	1·76	1·80	1·85	1·90	1·94
	8	1·86	1·82	1·79	1·76	1·73	1·73	1·73	1·76	1·79	1·82	1·86
	10	1·86	1·82	1·79	1·76	1·73	1·72	1·72	1·74	1·76	1·78	1·81
	15	1·86	1·82	1·79	1·76	1·73	1·71	1·71	1·71	1·72	1·73	1·75
	20	1·86	1·82	1·79	1·76	1·73	1·71	1·70	1·70	1·70	1·71	1·72
	∞	1·86	1·82	1·79	1·75	1·72	1·70	1·68	1·66	1·65	1·65	1·64
10	6	1·81	1·78	1·76	1·74	1·73	1·73	1·76	1·80	1·85	1·90	1·94
	8	1·81	1·78	1·76	1·74	1·72	1·72	1·73	1·76	1·79	1·82	1·86
	10	1·81	1·78	1·76	1·73	1·72	1·71	1·72	1·73	1·76	1·78	1·81
	15	1·81	1·78	1·76	1·73	1·72	1·70	1·70	1·71	1·72	1·73	1·75
	20	1·81	1·78	1·76	1·73	1·71	1·70	1·69	1·69	1·70	1·71	1·72
	∞	1·81	1·78	1·76	1·73	1·71	1·69	1·67	1·66	1·65	1·65	1·64
15	6	1·75	1·73	1·72	1·71	1·71	1·73	1·76	1·80	1·85	1·90	1·94
	8	1·75	1·73	1·72	1·71	1·71	1·71	1·73	1·76	1·79	1·82	1·86
	10	1·75	1·73	1·72	1·71	1·70	1·70	1·72	1·73	1·76	1·78	1·81
	15	1·75	1·73	1·72	1·70	1·70	1·69	1·70	1·70	1·72	1·73	1·75
	20	1·75	1·73	1·72	1·70	1·69	1·69	1·69	1·69	1·70	1·71	1·72
	∞	1·75	1·73	1·72	1·70	1·68	1·67	1·66	1·65	1·65	1·65	1·64
20	6	1·72	1·71	1·70	1·70	1·71	1·73	1·76	1·80	1·85	1·90	1·94
	8	1·72	1·71	1·70	1·70	1·70	1·71	1·73	1·76	1·79	1·82	1·86
	10	1·72	1·71	1·70	1·69	1·69	1·70	1·71	1·73	1·76	1·78	1·81
	15	1·72	1·71	1·70	1·69	1·69	1·69	1·69	1·70	1·72	1·73	1·75
	20	1·72	1·71	1·70	1·69	1·68	1·68	1·68	1·69	1·70	1·71	1·72
	∞	1·72	1·71	1·70	1·68	1·67	1·66	1·66	1·65	1·65	1·65	1·64
∞	6	1·64	1·65	1·66	1·67	1·69	1·72	1·76	1·80	1·85	1·90	1·94
	8	1·64	1·65	1·65	1·66	1·68	1·70	1·72	1·75	1·79	1·82	1·86
	10	1·64	1·65	1·65	1·66	1·67	1·69	1·71	1·73	1·76	1·78	1·81
	15	1·64	1·65	1·65	1·65	1·66	1·67	1·68	1·70	1·72	1·73	1·75
	20	1·64	1·65	1·65	1·65	1·66	1·66	1·67	1·68	1·70	1·71	1·72
	∞	1·64	1·64	1·64	1·64	1·64	1·64	1·64	1·64	1·64	1·64	1·64

* y is normally distributed about η with variance $\lambda_1\sigma_1^2 + \lambda_2\sigma_2^2$, and s_1^2 and s_2^2 are independent estimates of σ_1^2 and σ_2^2, based on f_1 and f_2 degrees of freedom, respectively. λ_1 and λ_2 are known constants.

In the problem of comparing the means of samples taken from two normal populations, put $y = (\bar{x}_1 - \bar{x}_2)$, $f_1 = (n_1 - 1)$, $f_2 = (n_2 - 1)$, $\lambda_1 = 1/n_1$ and $\lambda_2 = 1/n_2$, where n_1 and n_2 are the sample sizes.

From Pearson and Hartley, (1966), pp. 142-145.

Table B1 (*continued*)

Upper 2·5 % critical values of $v = \dfrac{(y-\eta)}{\sqrt{(\lambda_1 s_1^2 + \lambda_2 s_2^2)}}$

(*i.e. upper 5 % critical values of* $|v|$)

f_2	$\dfrac{\lambda_1 s_1^2}{(\lambda_1 s_1^2+\lambda_2 s_2^2)}$ f_1	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
8	8	2·31	2·25	2·20	2·14	2·10	2·08	2·10	2·14	2·20	2·25	2·31
	10	2·31	2·25	2·20	2·15	2·10	2·08	2·08	2·11	2·14	2·19	2·23
	12	2·31	2·25	2·20	2·15	2·10	2·07	2·07	2·08	2·11	2·14	2·18
	15	2·31	2·25	2·20	2·15	2·10	2·07	2·05	2·06	2·08	2·10	2·13
	20	2·31	2·25	2·20	2·15	2·10	2·06	2·04	2·04	2·05	2·07	2·09
	∞	2·31	2·25	2·20	2·14	2·09	2·05	2·01	1·99	1·97	1·96	1·96
10	8	2·23	2·19	2·14	2·11	2·08	2·08	2·10	2·15	2·20	2·25	2·31
	10	2·23	2·18	2·14	2·11	2·08	2·06	2·08	2·11	2·14	2·18	2·23
	12	2·23	2·18	2·14	2·10	2·07	2·06	2·06	2·08	2·11	2·14	2·18
	15	2·23	2·18	2·14	2·10	2·07	2·05	2·05	2·06	2·08	2·10	2·13
	20	2·23	2·18	2·14	2·10	2·07	2·05	2·04	2·04	2·05	2·06	2·09
	∞	2·23	2·18	2·14	2·10	2·06	2·03	2·00	1·98	1·97	1·96	1·96
12	8	2·18	2·14	2·11	2·08	2·07	2·07	2·10	2·15	2·20	2·25	2·31
	10	2·18	2·14	2·11	2·08	2·06	2·06	2·07	2·10	2·14	2·18	2·23
	12	2·18	2·14	2·11	2·08	2·06	2·05	2·06	2·08	2·11	2·14	2·18
	15	2·18	2·14	2·11	2·08	2·06	2·04	2·04	2·06	2·08	2·10	2·13
	20	2·18	2·14	2·11	2·08	2·05	2·04	2·03	2·03	2·05	2·06	2·09
	∞	2·18	2·14	2·11	2·07	2·04	2·02	1·99	1·98	1·97	1·96	1·96
15	8	2·13	2·10	2·08	2·06	2·05	2·07	2·10	2·15	2·20	2·25	2·31
	10	2·13	2·10	2·08	2·06	2·05	2·05	2·07	2·10	2·14	2·18	2·23
	12	2·13	2·10	2·08	2·06	2·04	2·04	2·06	2·08	2·11	2·14	2·18
	15	2·13	2·10	2·08	2·05	2·04	2·03	2·04	2·05	2·08	2·10	2·13
	20	2·13	2·10	2·08	2·05	2·04	2·03	2·03	2·03	2·05	2·06	2·09
	∞	2·13	2·10	2·07	2·05	2·02	2·00	1·99	1·97	1·97	1·96	1·96
20	8	2·09	2·07	2·05	2·04	2·04	2·06	2·10	2·15	2·20	2·25	2·31
	10	2·09	2·06	2·05	2·04	2·04	2·05	2·07	2·10	2·14	2·18	2·23
	12	2·09	2·06	2·05	2·03	2·03	2·04	2·05	2·08	2·11	2·14	2·18
	15	2·09	2·06	2·05	2·03	2·03	2·03	2·04	2·05	2·08	2·10	2·13
	20	2·09	2·06	2·05	2·03	2·02	2·02	2·02	2·03	2·05	2·06	2·09
	∞	2·09	2·06	2·04	2·02	2·01	1·99	1·98	1·97	1·96	1·96	1·96
∞	8	1·96	1·96	1·97	1·99	2·01	2·05	2·09	2·14	2·20	2·25	2·31
	10	1·96	1·96	1·97	1·98	2·00	2·03	2·06	2·10	2·14	2·18	2·23
	12	1·96	1·96	1·97	1·98	1·99	2·02	2·04	2·07	2·11	2·14	2·18
	15	1·96	1·96	1·97	1·97	1·99	2·00	2·02	2·05	2·07	2·10	2·13
	20	1·96	1·96	1·96	1·97	1·98	1·99	2·01	2·02	2·04	2·06	2·09
	∞	1·96	1·96	1·96	1·96	1·96	1·96	1·96	1·96	1·96	1·96	1·96

Table B1 (*continued*). *Test for comparisons involving two variances which must be separately estimated*

Upper 1% *critical values of* $v = \dfrac{(y-\eta)}{\sqrt{(\lambda_1 s_1^2 + \lambda_2 s_2^2)}}$

(*i.e. upper* 2% *critical values of* $|v|$)*

$\mathbf{f}_2$	$\dfrac{\lambda_1 s_1^2}{\lambda_1 s_1^2+\lambda_2 s_2^2}$ / f_1	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
10	10	2·76	2·70	2·63	2·56	2·51	2·50	2·51	2·56	2·63	2·70	2·76
	12	2·76	2·70	2·63	2·56	2·51	2·49	2·49	2·52	2·57	2·62	2·68
	15	2·76	2·70	2·63	2·56	2·51	2·48	2·47	2·48	2·52	2·56	2·60
	20	2·76	2·70	2·63	2·56	2·51	2·47	2·45	2·45	2·47	2·49	2·53
	30	2·76	2·70	2·63	2·56	2·50	2·46	2·43	2·42	2·42	2·44	2·46
	∞	2·76	2·70	2·63	2·56	2·50	2·44	2·40	2·36	2·34	2·33	2·33
12	10	2·68	2·62	2·57	2·52	2·49	2·49	2·51	2·56	2·63	2·70	2·76
	12	2·68	2·62	2·57	2·52	2·48	2·47	2·48	2·52	2·57	2·62	2·68
	15	2·68	2·62	2·57	2·52	2·48	2·46	2·46	2·48	2·52	2·56	2·60
	20	2·68	2·62	2·57	2·52	2·48	2·45	2·44	2·45	2·47	2·49	2·53
	30	2·68	2·62	2·57	2·52	2·47	2·44	2·42	2·41	2·42	2·44	2·46
	∞	2·68	2·62	2·57	2·51	2·46	2·42	2·38	2·36	2·34	2·33	2·33
15	10	2·60	2·56	2·52	2·48	2·47	2·48	2·51	2·56	2·63	2·70	2·76
	12	2·60	2·56	2·52	2·48	2·46	2·46	2·48	2·52	2·57	2·62	2·68
	15	2·60	2·56	2·51	2·48	2·45	2·45	2·45	2·48	2·51	2·56	2·60
	20	2·60	2·56	2·51	2·48	2·45	2·43	2·43	2·44	2·46	2·49	2·53
	30	2·60	2·56	2·51	2·47	2·44	2·42	2·41	2·41	2·42	2·44	2·46
	∞	2·60	2·56	2·51	2·47	2·43	2·40	2·37	2·35	2·34	2·33	2·33
20	10	2·53	2·49	2·47	2·45	2·45	2·47	2·51	2·56	2·63	2·70	2·76
	12	2·53	2·49	2·47	2·45	2·44	2·45	2·48	2·52	2·57	2·62	2·68
	15	2·53	2·49	2·46	2·44	2·43	2·43	2·45	2·48	2·51	2·56	2·60
	20	2·53	2·49	2·46	2·44	2·42	2·42	2·42	2·44	2·46	2·49	2·53
	30	2·53	2·49	2·46	2·44	2·42	2·40	2·40	2·40	2·42	2·43	2·46
	∞	2·53	2·49	2·46	2·43	2·40	2·38	2·36	2·34	2·33	2·33	2·33
30	10	2·46	2·44	2·42	2·42	2·43	2·46	2·50	2·56	2·63	2·70	2·76
	12	2·46	2·44	2·42	2·41	2·42	2·44	2·47	2·52	2·57	2·62	2·68
	15	2·46	2·44	2·42	2·41	2·41	2·42	2·44	2·47	2·51	2·56	2·60
	20	2·46	2·43	2·42	2·40	2·40	2·40	2·42	2·44	2·46	2·49	2·53
	30	2·46	2·43	2·42	2·40	2·39	2·39	2·39	2·40	2·42	2·43	2·46
	∞	2·46	2·43	2·41	2·39	2·37	2·36	2·35	2·34	2·33	2·33	2·33
∞	10	2·33	2·33	2·34	2·36	2·40	2·44	2·50	2·56	2·63	2·70	2·76
	12	2·33	2·33	2·34	2·36	2·38	2·42	2·46	2·51	2·57	2·62	2·68
	15	2·33	2·33	2·34	2·35	2·37	2·40	2·43	2·47	2·51	2·56	2·60
	20	2·33	2·33	2·33	2·34	2·36	2·38	2·40	2·43	2·46	2·49	2·53
	30	2·33	2·33	2·33	2·34	2·35	2·36	2·37	2·39	2·41	2·43	2·46
	∞	2·33	2·33	2·33	2·33	2·33	2·33	2·33	2·33	2·33	2·33	2·33

* y is normally distributed about η with variance $\lambda_1\sigma_1^2+\lambda_2\sigma_2^2$, and s_1^2 and s_2^2 are independent estimates of σ_1^2 and σ_2^2, based on $\mathbf{f}_1$ and $\mathbf{f}_2$ degrees of freedom, respectively. λ_1 and λ_2 are known constants.

In the problem of comparing the means of samples taken from two normal populations, put $y=(\bar{x}_1-\bar{x}_2)$, $\mathbf{f}_1=(n_1-1)$, $\mathbf{f}_2=(n_2-1)$, $\lambda_1=1/n_1$ and $\lambda_2=1/n_2$, where n_1 and n_2 are the sample sizes.

Table B1 (*continued*)

Upper 0·5 % critical values of $v = \dfrac{y-\eta}{\sqrt{(\lambda_1 s_1^2 + \lambda_2 s_2^2)}}$

(*i.e. upper* 1 % *critical values of* $|v|$)

	$\dfrac{\lambda_1 s_1^2}{(\lambda_1 s_1^2+\lambda_2 s_2^2)}$	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
f_2	f_1											
10	10	3·17	3·08	3·00	2·90	2·82	2·79	2·82	2·90	3·00	3·08	3·17
	12	3·17	3·08	3·00	2·91	2·82	2·78	2·79	2·84	2·91	2·98	3·05
	15	3·17	3·08	3·00	2·91	2·82	2·77	2·76	2·78	2·83	2·89	2·95
	20	3·17	3·08	3·00	2·91	2·82	2·76	2·73	2·74	2·76	2·80	2·85
	30	3·17	3·08	3·00	2·91	2·82	2·75	2·71	2·69	2·70	2·72	2·75
	∞	3·17	3·08	2·99	2·91	2·82	2·74	2·67	2·63	2·60	2·58	2·58
12	10	3·05	2·98	2·91	2·84	2·79	2·78	2·82	2·91	3·00	3·08	3·17
	12	3·05	2·98	2·91	2·84	2·78	2·76	2·78	2·84	2·91	2·98	3·05
	15	3·05	2·98	2·91	2·84	2·78	2·75	2·75	2·78	2·83	2·89	2·95
	20	3·05	2·98	2·91	2·84	2·78	2·74	2·72	2·73	2·76	2·80	2·85
	30	3·05	2·98	2·91	2·84	2·77	2·73	2·70	2·69	2·70	2·72	2·75
	∞	3·05	2·98	2·91	2·84	2·77	2·71	2·65	2·62	2·59	2·58	2·58
15	10	2·95	2·89	2·83	2·78	2·76	2·77	2·82	2·91	3·00	3·08	3·17
	12	2·95	2·89	2·83	2·78	2·75	2·75	2·78	2·84	2·91	2·98	3·05
	15	2·95	2·89	2·83	2·78	2·74	2·73	2·74	2·78	2·83	2·89	2·95
	20	2·95	2·89	2·83	2·78	2·74	2·71	2·71	2·73	2·76	2·80	2·85
	30	2·95	2·89	2·83	2·78	2·73	2·70	2·68	2·68	2·70	2·72	2·75
	∞	2·95	2·89	2·83	2·77	2·72	2·67	2·64	2·61	2·59	2·58	2·58
20	10	2·85	2·80	2·76	2·74	2·73	2·76	2·82	2·91	3·00	3·08	3·17
	12	2·85	2·80	2·76	2·73	2·72	2·74	2·78	2·84	2·91	2·98	3·05
	15	2·85	2·80	2·76	2·73	2·71	2·71	2·74	2·78	2·83	2·89	2·95
	20	2·85	2·80	2·76	2·73	2·70	2·70	2·70	2·73	2·76	2·80	2·85
	30	2·85	2·80	2·76	2·72	2·69	2·68	2·67	2·68	2·70	2·72	2·75
	∞	2·85	2·80	2·76	2·72	2·68	2·65	2·62	2·60	2·59	2·58	2·58
30	10	2·75	2·72	2·70	2·69	2·71	2·75	2·82	2·91	3·00	3·08	3·17
	12	2·75	2·72	2·70	2·69	2·70	2·73	2·77	2·84	2·91	2·98	3·05
	15	2·75	2·72	2·70	2·68	2·68	2·70	2·73	2·78	2·83	2·89	2·95
	20	2·75	2·72	2·70	2·68	2·67	2·68	2·69	2·72	2·76	2·80	2·85
	30	2·75	2·72	2·69	2·67	2·66	2·66	2·66	2·67	2·69	2·72	2·75
	∞	2·75	2·72	2·69	2·66	2·64	2·62	2·60	2·59	2·58	2·58	2·58
∞	10	2·58	2·58	2·60	2·63	2·67	2·74	2·82	2·91	2·99	3·08	3·17
	12	2·58	2·58	2·59	2·62	2·65	2·71	2·77	2·84	2·91	2·98	3·05
	15	2·58	2·58	2·59	2·61	2·64	2·67	2·72	2·77	2·83	2·89	2·95
	20	2·58	2·58	2·59	2·60	2·62	2·65	2·68	2·72	2·76	2·80	2·85
	30	2·58	2·58	2·58	2·59	2·60	2·62	2·64	2·66	2·69	2·72	2·75
	∞	2·58	2·58	2·58	2·58	2·58	2·58	2·58	2·58	2·58	2·58	2·58